GVSDEPCAPX - Modeling Fixed Assets

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The balance of fixed assets at any future time t is a function of the balance of fixed assets at time zero (i.e. today) plus capital expenditures over the time interval $[0,t]$ minus depreciation over the time interval $[0,t]$. In this white paper we will develop the mathematics to model fixed assets, depreciation expense and capital expenditures over time. To assist us in this endeavor we will work through the following hypothetical problem...

Our Hypothetical Problem

We are tasked with building a model for fixed assets given the following model parameters...

Table 1: Model Parameters (Amounts are in dollars)

Description	Current Year	Prior Year
Total revenue for the year	50,000,000	48,000,000
Fixed assets (net) end of year	15,100,000	14,300,000
Depreciation expense for the year	4,500,000	–

Our Problem: Reconcile the change in fixed assets in year five.

Annualized Revenue

We will define the variable R_t to be annualized revenue at time t and the variable μ to be the continuous-time revenue growth rate. The equation for annualized revenue at time t as a function of annualized revenue at time zero is...

$$R_t = R_0 \,\mathrm{Exp}\left\{\mu\, t\right\} \text{ ...where... } \frac{\delta R_t}{\delta t} = \mu\, R_0 \,\mathrm{Exp}\left\{\mu\, t\right\} = \mu\, R_t \tag{1}$$

We will define the variable $R_{a,b}$ to be cumulative revenue realized over the time interval $[a,b]$. Using Equation (1) above the equation for cumulative revenue is...

$$R_{a,b} = \int\limits_a^b R_t\,\delta t = \int\limits_a^b R_0 \,\mathrm{Exp}\left\{\mu\, t\right\}\delta t = R_0 \int\limits_a^b \mathrm{Exp}\left\{\mu\, t\right\}\delta t \tag{2}$$

Using Appendix Equation (26) below the solution to Equation (2) above is...

$$R_{a,b} = R_0\left(\mathrm{Exp}\left\{\mu\, b\right\} - \mathrm{Exp}\left\{\mu\, a\right\}\right)\mu^{-1} \tag{3}$$

Note that for the equations that follow we will need an estimate of annualized revenue at time zero. If we are given total revenue for the current period and the revenue growth rate over that period then using Equation (3) above we can solve for annualized revenue at time zero as follows...

$$R_0 = \frac{\mu\, R_{a,b}}{\mathrm{Exp}\left\{\mu\, b\right\} - \mathrm{Exp}\left\{\mu\, a\right\}} \text{ ...where... } a = \text{beginning of period ...and... } b = \text{end of period} \tag{4}$$

Fixed Assets

We will define fixed assets to be property, plant and equipment net of accumulated depreciation. For our purposes we will include only depreciable assets in our definition of fixed assets and accordingly will exclude non-depreciable assets such as land. The equation for fixed assets at time b is fixed assets at time a plus capital expenditures over the time interval $[a,b]$ minus depreciation over the time interval $[a,b]$. This statement in equation form is...

$$F_b = F_a + \text{Capital expenditures over } [a,b] - \text{Depreciation expense over } [a,b] \tag{5}$$

We will model fixed assets at time t to be a function of annualized revenue at time t. We will define the variable F_t to be fixed assets at time t and the variable θ to be the constant ratio of fixed assets to annualized revenue. Using Equation (1) above the equation for fixed assets is...

$$F_t = \theta R_t = \theta R_0 \operatorname{Exp}\left\{\mu\, t\right\} \tag{6}$$

We will define the variable $D_{a,b}$ to be depreciation expense realized over the time interval $[a,b]$ and the variable λ to be the continuous-time depreciation rate. Using Equation (6) above the equation for depreciation expense is...

$$D_{a,b} = \int\limits_a^b \lambda\, F_t\, \delta t = \lambda\, \theta R_0 \int\limits_a^b \operatorname{Exp}\left\{\mu\, t\right\} \delta t \tag{7}$$

Using Appendix Equation (26) below the solution to Equation (7) above when $\mu \neq 0$ is...

$$D_{a,b} = \frac{\lambda\, \theta}{\mu}\, R_0 \left(\operatorname{Exp}\left\{\mu\, b\right\} - \operatorname{Exp}\left\{\mu\, a\right\}\right) \text{ ...when... } \mu \neq 0 \tag{8}$$

Note that the solution to Equation (7) above when $\mu = 0$ is...

$$D_{a,b} = \lambda\, \theta\, R_0\, (b-a) \text{ ...when... } \mu = 0 \tag{9}$$

Capital expenditures are defined as the increase in fixed assets needed to support the increase in annualized revenue plus fixed asset replacement due to depreciation. Using Equation (5) above the general equation for capital expenditures is...

$$\text{Capital expenditures over } [a,b] = F_b - F_a + \text{Depreciation expense over } [a,b] \tag{10}$$

We will define the variable $X_{a,b}$ to be capital expenditures realized over the time interval $[a,b]$.

$$X_{a,b} = \theta \int\limits_a^b \frac{\delta R_t}{\delta t}\, \delta t + \lambda\, \theta R_0 \int\limits_a^b \operatorname{Exp}\left\{\mu\, t\right\} \delta t \tag{11}$$

Using Equation (8) above and Appendix Equation (28) below the solution to Equation (11) above when $\mu \neq 0$ is...

$$\begin{aligned} X_{a,b} &= \theta\, R_0 \left(\operatorname{Exp}\left\{\mu\, b\right\} - \operatorname{Exp}\left\{\mu\, a\right\}\right) + \frac{\lambda\, \theta}{\mu}\, R_0 \left(\operatorname{Exp}\left\{\mu\, b\right\} - \operatorname{Exp}\left\{\mu\, a\right\}\right) \\ &= \theta\, R_0 \left(1 + \frac{\lambda}{\mu}\right) \left(\operatorname{Exp}\left\{\mu\, b\right\} - \operatorname{Exp}\left\{\mu\, a\right\}\right) \\ &= \theta\, R_0 \left(\frac{\mu + \lambda}{\mu}\right) \left(\operatorname{Exp}\left\{\mu\, b\right\} - \operatorname{Exp}\left\{\mu\, a\right\}\right) \text{ ...when... } \mu \neq 0 \end{aligned} \tag{12}$$

Note that the solution to Equation (11) above when $\mu = 0$ is...

$$X_{a,b} = D_{a,b} \text{ ...when... } \mu = 0 \tag{13}$$

We will define the variable ω to be the weighted average remaining life of fixed assets. We can get an estimate of remaining average life via the following equation...

$$\omega = \frac{\text{Average fixed assets balance over the period}}{\text{Depreciation expense for the period}} \tag{14}$$

Using Equation (14) above the value of the parameter λ (continuous-time depreciation rate) as a function of the parameter estimate for ω (weighted average life) above is... [1]

$$\omega = \int\limits_0^\infty t\, \lambda\, F_t\, \delta t \div \int\limits_0^\infty \lambda\, F_t\, \delta t = \frac{1}{\lambda} \text{ ...such that... } \lambda = \frac{1}{\omega} \tag{15}$$

Model Parameter Estimates

We defined the variable μ to be the continuous-time revenue growth rate. Using Equation (1) above the equation for the continuous-time revenue growth rate is...

$$\text{if... } R_b = R_a \operatorname{Exp}\left\{\mu\,(b-a)\right\} \text{ ...then... } \mu = \frac{\ln(R_b) - \ln(R_b)}{b-a} \tag{16}$$

Using Equation (16) above and the data in Table 1 above the revenue growth rate over the time interval $[-1, 0]$ is...

$$\mu = \frac{\ln(50,000,000) - \ln(48,000,000)}{1} = 0.0408 \tag{17}$$

Using Equations (4) and (17) above and the data in Table 1 above annualized revenue at time zero is...

$$R_0 = \frac{0.0408 \times 50,000,000}{\operatorname{Exp}\{0.0408 \times 0\} - \operatorname{Exp}\{0.0408 \times -1\}} = 51,027,000 \tag{18}$$

Using Equation (14) above and the data in Table 1 above the estimated value of the parameter ω (weighted average remaining fixed assets life) is...

$$\omega = \frac{(15,100,000 + 14,300,000) \div 2}{4,500,000} = 3.2667 \tag{19}$$

Using Equations (15) and (19) above the estimated value of the parameter λ (continuous-time depreciation rate) is...

$$\lambda = \frac{1}{3.2667} = 0.3061 \tag{20}$$

Using Equations (6) and (18) above and the data in Table 1 above the estimated value of the parameter θ (ratio of fixed assets (net) to annualized revenue) is...

$$\theta = \frac{15,100,000}{51,027,000} = 0.2959 \tag{21}$$

The Solution To Our Hypothetical Problem

Using Equation (6) above and the paramter estimates above the balance of fixed assets at the end of year four is...

$$F_4 = 0.2959 \times 51,027,000 \times \operatorname{Exp}\left\{0.0408 \times 4\right\} = 17,778,370 \tag{22}$$

Using Equation (6) above and the paramter estimates above the balance of fixed assets at the end of year five is...

$$F_5 = 0.2959 \times 51,027,000 \times \operatorname{Exp}\left\{0.0408 \times 5\right\} = 18,519,140 \tag{23}$$

Using Equation (8) above and the paramter estimates above depreciation expense in year five is...

$$D_{4,5} = \frac{0.3061 \times 0.2959}{0.0408} \times 51,027,000 \times \left(\operatorname{Exp}\left\{0.0408 \times 5\right\} - \operatorname{Exp}\left\{0.0408 \times 4\right\}\right) = 5,554,970 \tag{24}$$

Using Equation (12) above and the paramter estimates above capital expenditures in year five is...

$$X_{4,5} = 0.2959 \times 51,027,000 \times \left(\frac{0.0408 + 0.3061}{0.0408}\right)\left(\operatorname{Exp}\left\{0.0408 \times 5\right\} - \operatorname{Exp}\left\{0.0408 \times 4\right\}\right) = 6,295,740 \tag{25}$$

The solution to our hypothetical problem is...

Description	Balance	Reference
Beginning balance	17,778,370	Equation (22)
Capital expenditures	6,295,740	Equation (25)
Depreciation expense	-5,554,970	Equation (24)
Ending balance	18,519,140	Equation (23)

References

[1] Gary Schurman, *Integration By Parts - Part II - Weighted Average Life,* January, 2020

Appendix

A. The solution to the following integral is...

$$\int\limits_a^b \mathrm{Exp}\left\{m\,t\right\}\delta t = \frac{1}{m}\,\mathrm{Exp}\left\{m\,t\right\}\Bigg[_a^b = \frac{1}{m}\left(\mathrm{Exp}\left\{m\,b\right\} - \mathrm{Exp}\left\{m\,a\right\}\right) \tag{26}$$

B. The solution to the following integral is...

$$\int\limits_a^b \frac{\delta R_t}{\delta t}\,\delta t = R_b - R_a \tag{27}$$

Using Equation (1) above we can rewrite Appendix Equation (27) above as...

$$\int\limits_a^b \frac{\delta R_t}{\delta t}\,\delta t = R_0\left(\mathrm{Exp}\left\{\mu\,b\right\} - \mathrm{Exp}\left\{\mu\,a\right\}\right) \tag{28}$$